ISRAEL JOURNAL OF MATHEMATICS **138** (2003), 61–72

SYMMETRIC PRODUCTS OF SURFACES AND THE CYCLE INDEX*

BY

PAVLE BLAGOJEVIĆ

Mathematics Institute SANU, Knez Mihailova 35/1, P.F. 367
11001 Belgrade, Yugoslavia
e-mail: vxdig@eunet.yu

AND

VLADIMIR GRUJIĆ

Faculty of Mathematics, Studentski TRG 16
11000 Belgrade, Yugoslavia
e-mail: vgrujic@matf.bg.ac.yu

AND

RADE ŽIVALJEVIĆ

Mathematics Institute SANU, Knez Mihailova 35/1, P.F. 367
11001 Belgrade, Yugoslavia
e-mail: rade@turing.mi.sanu.ac.yu

ABSTRACT

We study some of the combinatorial structures related to the signature of G-symmetric products of (open) surfaces $SP^m_G(M) = M^m/G$ where $G \subset S_m$. The attention is focused on the question, what information about a surface M can be recovered from a symmetric product $SP^n(M)$. The problem is motivated in part by the study of locally Euclidean topological commutative $(m+k, m)$-groups, [16]. Emphasizing a combinatorial point of view we express the signature $\mathrm{Sign}(SP^m_G(M))$ in terms of the cycle index $Z(G; \bar{x})$ of G, a polynomial which originally appeared in Pólya enumeration theory of graphs, trees, chemical structures etc. The computations are used to show that there exist punctured Riemann surfaces $M_{g,k}, M_{g',k'}$ such that the manifolds $SP^m(M_{g,k})$ and $SP^m(M_{g',k'})$ are often not homeomorphic, although they always have the same homotopy type provided $2g + k = 2g' + k'$ and $k, k' \geq 1$.

* Supported by the Serbian Ministry for Science and Technology, Grant No. 1643.
Received September 3, 2001 and in revised form September 19, 2002

1. Introduction

The complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and the elliptic curves are classical examples of surfaces that support a group structure. A natural generalization of the (commutative) group structure is the structure of a (commutative) $(m+k, m)$-group.

Let $SP^n(X) := X^n/S_n$ be the symmetric product of X and let

$$SP^p(X) \times SP^q(X) \to SP^{p+q}(X), \quad (a, b) \mapsto c := a * b$$

be the operation induced by concatenation of strings $a \in SP^p(X)$ and $b \in SP^q(X)$. A **commutative** $(m+k, m)$-groupoid is a pair (X, μ) where the "multiplication" μ is an arbitrary map μ: $SP^{m+k}(X) \to SP^m(X)$. The operation μ is associative if for each $c \in SP^{m+2k}(X)$ and each representation $c = a * b$, where $a \in SP^{m+k}(X)$ and $b \in SP^k(X)$, the result $\mu(\mu(a) * b)$ is always the same, i.e., independent of the particular choice of a and b in the representation $c = a * b$. A commutative and associative $(m+k, m)$-groupoid is a $(m+k, m)$-group if the equation $\mu(x * a) = b$ has a solution $x \in SP^m(X)$ for each $a \in SP^k(X)$ and $b \in SP^m(X)$. Note that $(2, 1)$-groups are essentially the groups in the usual sense of the word. If X is a topological space, then (X, μ) is a topological $(m+k, m)$-group if it is a $(m+k, m)$-group and the map μ: $SP^{m+k}(X) \to SP^m(X)$ is continuous.

For the motivation and other information about commutative $(m+k, m)$-groups the reader is referred to [16], [15]. Surprisingly enough, the only known surfaces that support the structure of a $(m+k, m)$-group for $(m+k, m) \neq (2, 1)$ are of the form $\mathbb{C} \setminus A$ where A is a finite set. Moreover, it was proved in [15], see also Theorem 6.1 in [16], that if (M, μ) is a locally Euclidean, topological, commutative, $(m+k, m)$-group then M must be an orientable 2-manifold. Moreover, a 2-manifold that admits the structure of a commutative $(m+k, m)$-group satisfies a strong necessary condition that the symmetric power $SP^m(M) := M^m/S_m$ is of the form $\mathbb{R}^u \times (S^1)^v$. In particular, the signature $\mathrm{Sign}(SP^m(M))$ of M must be zero.

There is a conjecture [16] that the only examples of topological, commutative, $(m+k, m)$-groups are supported by surfaces of the form $M = \mathbb{C} \setminus A$. Corollary 1.3 and Proposition 1.4 support this conjecture, since they imply that all open surfaces of sufficiently high genus have a non-zero signature. However, this conjecture serves also as a partial motivation for the following general questions which may be of some independent interest.

Questions:

(A) To what extent is the topology of a surface M determined by the topology of its symmetric product $SP^m(M)$ for a given m?

(B) Are there examples of non-homeomorphic (open) surfaces M and N such that the associated symmetric products $SP^m(M)$ and $SP^m(N)$ are homeomorphic?

In response to (A) we prove the following theorem which says that homological invariants alone are not sufficient to distinguish symmetric products of non-homeomorphic surfaces. This puts some limitations on surfaces M and N in question (B), however the question itself remains open and interesting already in the case of general surfaces $M_{g,k}$.

THEOREM 1.1: *There exist open, orientable surfaces M and N such that the associated symmetric products $SP^m(M)$ and $SP^m(N)$ are not homeomorphic although they have the same homotopy type. More precisely, this is always true if $M = M_{g,k}$ and $N = M_{g',k'}$ $(k, k' \geq 1)$ and*

- $2g + k = 2g' + k'$,
- $g \neq g'$ *and* $\max\{g, g'\} \geq m/2$

where $M_{g,k} := M_g \setminus \{x_1, \dots, x_k\}$ is the surface of genus g punctured at k points.

A natural approach to questions (A) and (B) is to determine what information about M is hidden in $SP^m(M)$. If $SP^m(M)$ is known, then the iterated symmetric product $SP^k(SP^m(M))$ and its higher order analogs are also known. Since $SP^k(SP^m(M)) \cong SP_G(M) := M^{mk}/G$ where $G = S_k \wr S_m$ is the *wreath* product of groups, it is natural to consider general G-symmetric products $SP_G(M) = M^N/G$ where $G \subset S_N$ is an arbitrary subgroup of S_N. When we want to emphasize that G is a subgroup of S_N we write $SP_G(M) = SP_G^N(M)$. $SP_G(M)$ is always a $\mathbb{Q}$-homology manifold and the signature $\mathrm{Sign}(SP_G(M))$ is well defined. Our central technical result is the following theorem.

THEOREM 1.2:

$$\mathrm{Sign}(SP_G^m(M_{g,k})) = Z(G; 0, -2g, 0, -2g, \dots) \tag{1}$$

where $G \subset S_m$ and $Z(G; x_1, x_2, \dots, x_m)$ is the **cycle index** *of G.*

COROLLARY 1.3:

$$\mathrm{Sign}(SP^{2n}(M_{g,k})) = (-1)^n \binom{g}{n}. \tag{2}$$

Recall that the cycle index, [1], [11], [3], [14], is a fundamental polynomial which was originally defined by J. H. Redfield and independently by G. Pólya who recognized its central role in his celebrated enumeration theory of groups, graphs, trees, chemical compounds etc. The advantage of expressing the result in terms of the cycle index lies in the fact that the cycle index is a well studied object of enumerative combinatorics. One of highlights is Theorem 1.16, see [1] Section V, which describes a procedure how the cycle index $Z(G;\vec{x})$ of the wreath product $G = Q \wr H$ can be in a transparent and elegant way expressed in terms of cycle indices $Z(Q;\vec{x}), Z(H;\vec{x})$ of Q and H respectively. A good illustration how these ideas can be applied is provided by the following proposition which itself can be seen as another corollary of Theorem 1.2.

PROPOSITION 1.4: *Suppose that m is odd and p an even integer. Then*

$$\mathrm{Sign}(SP_{S_p \wr S_m}(M_{g,k})) = Z(S_p \wr S_m; 0, -2g, \ldots, 0, -2g) = (-1)^{p/2}\binom{\frac{1}{2}\binom{2g}{m}}{\frac{1}{2}p}. \tag{3}$$

COROLLARY 1.5: *If m is an odd integer and $k \geq 1$ then*

$$\mathrm{Sign}(SP^2(SP^m(M_{g,k}))) = -\frac{1}{2}\binom{2g}{m}. \tag{4}$$

For completeness we recall a remarkable formula of Don Zagier* obtained by the Atiyah–Singer G-signature theorem applied to (σ_m, M^m), where $\sigma\colon M^m \to M^m$ is the cyclic permutation.

THEOREM 1.6 (Zagier [17]):

$$\begin{aligned}\mathrm{Sign}(SP^m_G(M)) &= Z(G; \tau, \chi, \tau, \chi, \ldots)\\ &= Z(G;\vec{x})|_{x_{2i}=\chi, x_{2i-1}=\tau}\end{aligned} \tag{5}$$

where $\tau = \tau(M)$ and $\chi = \chi(M)$ are respectively the signature and the Euler characteristic of a compact, oriented, even-dimensional manifold without boundary.

Again the use of the cycle index is convenient. In the case $G = S_k \wr S_m$ one obtains, along the lines of Proposition 1.4, the following result which for $m = 1$ reduces to the formula of Hirzebruch, [8], [17]. By convention [14], $[t^p]f(t)$ is the coefficient of t^p in the power series $f(t)$.

---

* Zagier did not originally express the result in terms of the cycle index $Z(G;\bar{x})$. His result says that $\mathrm{Sign}(\sigma_m, M^m)$ is either $\tau(M)$ or $\chi(M)$ depending on the parity of m.

THEOREM 1.7:

$$\text{(6)}\qquad \operatorname{Sign}(SP_{S_k \wr S_m}(M)) = [t^k](1-t^2)^{\frac{1}{2}(-1)^{m+1}\binom{-\chi}{m}} \left(\frac{1+t}{1-t}\right)^{\frac{1}{2}\operatorname{Sign}(SP^m(M))}.$$

1.1 SIGNATURE AS A FUNCTION OF BOTH G AND g. Our proof of Theorem 1.2 with minor modification yields a proof of the corresponding well known statement for closed surfaces M_g. This allows us to check our computations, so we find it convenient to formulate and prove these two results as parts of a single statement.

THEOREM 1.8:

$$\text{(7)}\qquad \begin{aligned} \text{(a)}\quad \operatorname{Sign}(SP^m_G(M_g)) =& Z(G; 0, 2-2g, 0, 2-2g, \dots) \\ =& Z(G;\bar{x})|_{x_{2i}=2-2g, x_{2i-1}=0}, \\ \text{(b)}\quad \operatorname{Sign}(SP^m_G(M_{g,k})) =& Z(G; 0, -2g, 0, -2g, \dots) \\ =& Z(G;\bar{x})|_{x_{2i}=-2g, x_{2i-1}=0}, \end{aligned}$$

where $Z(G; x_1, \dots, x_m)$ is the cycle index of the permutation group $G \subset S_m$.

Before we commence the proof, let us recall some generalities about the g-signature of G-manifolds or vector spaces with G-invariant bilinear forms, [9], [4].

Let V be a vector space, $B: V \times V \to \mathbb{C}$ a hermitian bilinear form on V and $g: V \to V$ an endomorphism which preserves the form B, $B(gx, gy) = B(x, y)$ for all $x, y \in V$. Then V admits a g-invariant decomposition $V \cong V^0 \oplus V^+ \oplus V^-$ such that B is positive definite on V^+, negative definite on V^- and zero on V^0. Then the g signature of the triple (V, B, g), or for short the g-signature of V, is defined by

$$\operatorname{Sign}(g, V) = \operatorname{Sign}(g, V, B) = \operatorname{Trace}(g|V^+) - \operatorname{Trace}(g|V^-).$$

A symmetric or skew-symmetric form $B: V \times V \to \mathbb{R}$ defined on a real vector space V can be extended to a hermitian form $\widehat{B}$ on the complexified space $V \otimes \mathbb{C}$ by the formula [4]

$$\text{(8)}\qquad \widehat{B}(x \otimes \alpha, y \otimes \beta) = \begin{cases} \alpha\bar{\beta}B(x,y), & B \text{ is symmetric,} \\ i\alpha\bar{\beta}B(x,y), & B \text{ is skew-symmetric.} \end{cases}$$

The associated g-signature is also denoted by $\operatorname{Sign}(g, V)$. Finally, suppose that M^{2n} is a smooth, oriented manifold, with or without boundary, and let $g: M^{2n} \to M^{2n}$ be an orientation preserving diffeomorphism of M. The intersection form $B: H_n(M, \mathbb{Q}) \times H_n(M, \mathbb{Q}) \to \mathbb{Q}$ is symmetric if n is even, or skew-symmetric if n is odd, and the associated endomorphism $\tilde{g} := H_n(g)\colon H_n(M) \to H_n(M)$ preserves both the intersection form B and its complexification. The g-signature of $(\tilde{g}, V, B)$ is in this case denoted by $\mathrm{Sign}(g, M)$. In particular, we observe that the usual signature $\mathrm{Sign}(M)$ can be interpreted as the g-signature $\mathrm{Sign}(\mathrm{Id}, M)$ of the identity map Id: $M \to M$.

The following well known result, [6], [9], is of fundamental importance. Note that even the case of 0-dimensional manifolds (finite sets) is interesting, when this result reduces to an elementary lemma (Burnside lemma) which is a corner-stone of Pólya enumeration theory.

PROPOSITION 1.9: *Suppose that M^{2n} is a smooth, oriented manifold with a not necessarily free, orientation preserving action of a finite group G of diffeomorphism. More generally, it is sufficient to assume that M is a $\mathbb{Q}$-homology manifold. Then M/G is a $\mathbb{Q}$-homology manifold,* $\mathrm{Sign}(M/G)$ *is well defined and the following formula holds,* [6], [9],

$$\mathrm{Sign}(M/G) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Sign}(g, M).$$

The following proposition is used in the proof of Theorem 1.8.

PROPOSITION 1.10: *Let V be a $(2n)$-dimensional complex vector space and let $B\colon V \times V \to \mathbb{C}$ be a hermitian form. Suppose $\omega\colon V \to V$ is an endomorphism preserving the form B, such that $\omega^{2n} = 1$, and for some $v_0 \in V$, the set $\{v_0, \omega(v_0), \dots, \omega^{2n-1}(v_0)\}$ is a basis of V. Suppose that $B(v_0, \omega^j(v_0)) \neq 0 \Longrightarrow j = n$. Then,*

$$\begin{aligned}\mathrm{Sign}(\omega, V) &= \Big(\sum_{x^{2n}=1} x^{n+1}\Big) signB(v_0, \omega^n(v_0)) \\ &= \begin{cases} 0, & n \neq 1, \\ 2signB(v_0, \omega(v_0)), & n = 1.\end{cases}\end{aligned} \tag{9}$$

Proof: Let $x \in \{1, \epsilon, \epsilon^2, \dots, \epsilon^{2n-1}\}$ be a root of unity, $\epsilon = e^{2\pi i/2n}$. Let $z_x := v_0 + x^{-1}\omega(v_0) + x^{-2}\omega^2(v_0) + \cdots + x^{-(2n-1)}\omega^{2n-1}(v_0)$ be the eigenvector of ω corresponding to the eigenvalue x. If x and y are different eigenvalues, $B(z_x, z_y) = B(\omega(z_x), \omega(z_y)) = x\bar{y}B(z_x, z_y) = 0$. Otherwise, since $B(v_0, \omega^j(v_0))$ can be nonzero only for $j = n$, we have $B(z_x, z_x) = 2nx^nB(v_0, \omega^n(v_0))$. By a well known formula,

$$\mathrm{Sign}(V, B) = \sum_{\lambda\in\mathrm{Spec}(\omega)} \lambda\,\mathrm{Sign}(V_\lambda, B_\lambda),$$

where V_λ is the eigenspace of ω which corresponds to the eigenvalue λ and $B_\lambda = B|V_\lambda$ is the restriction of the form B on V_λ. In our case,

$$\mathrm{Sign}(V, B) = \sum_{x^{2n}=1} x \; signB(z_x, z_x) = (\sum_{x^{2n}=1} x^{n+1}) signB(v_0, \omega^n(v_0))$$

and equation (9) follows. ∎

Our next step in the preparations for the proof of Theorem 1.8 is an explicit description of the intersection pairing $B: H_m(M_g^m;\mathbb{Q}) \times H_m(M_g^m;\mathbb{Q}) \to \mathbb{Q}$. Note that both sides of the equations (7) are zero if m is an odd number. So from here on, we focus our attention on the even case and assume that $m = 2n$.

Let us choose an orientation on M_g and let $\mathbb{T} \in H_2(M_g;\mathbb{Q})$ be the associated fundamental class. Let $a_1, b_1, \dots, a_g, b_g$ be a symplectic basis of $H_1(M_g)$ so that $a_i \cap a_i = b_j \cap b_j = 0, a_i \cap b_j = 0$ for $i \neq j$, and $a_i \cap b_i = \mathbb{I}$, where $\mathbb{I} \in H_0(M_g)$ is the generator. By the Künneth formula,

$$H_{2n}(M_g^{2n};\mathbb{Q}) \cong \bigotimes_{k_1+k_2+\cdots+k_{2n}=2n} H_{k_i}(M_g;\mathbb{Q}).$$

From here one deduces that a convenient basis of $H_{2n}(M_g^{2n};\mathbb{Q})$ consists of all "words" $w = w_1 w_2 \cdots w_{2n} = w_1 \times w_2 \times \cdots \times w_{2n}$ where

$$w_k \in \{a_i\}_{i=1}^g \cup \{b_j\}_{j=1}^g \cup \{\mathbb{I}, \mathbb{T}\}.$$

Note that for dimensional reasons, the number of occurrences of the letter $\mathbb{I}$ in the word w is equal to the number of occurrences of the letter $\mathbb{T}$.

LEMMA 1.11: *Let $w = w_1 w_2 \cdots w_{2n}$ and $w' = w'_1 w'_2 \cdots w'_{2n}$ be two words, representing basic homology classes in $H_{2n}(M_g^{2n})$. Then,*

$$B(w, w') = \epsilon_{w,w'} \langle w_1, w'_1\rangle \cdots \langle w_{2n}, w'_{2n}\rangle \tag{10}$$

where $\epsilon_{w,w'}$ is either $+1$ or -1, while $B(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are the intersection pairings on groups $H_{2n}(M_g^{2n})$ and $H_1(M_g)$ respectively.

Proof: Indeed, $w = w_1 \times \cdots \times w_{2n} = \widehat{w}_1 \cap \cdots \cap \widehat{w}_{2n}$ where,

$$\widehat{w}_i = \mathbb{T} \times \cdots \times w_i \times \cdots \times \mathbb{T} \in H_*(M_g^{2n}).$$

Hence, $B(w_1 \times \cdots \times w_{2n}, w'_1 \times \cdots \times w'_{2n}) = (\widehat{w}'_1 \cap \cdots \cap \widehat{w}'_{2n}) \cap (\widehat{w}_1 \cap \cdots \cap \widehat{w}_{2n})$ and it is sufficient to remember that

$$a \cap b = (-1)^{\mathrm{cd}(a)\mathrm{cd}(b)}\, b \cap a$$

where $\mathrm{cd}(x)$ is the codimension of a class x. ∎

Let us define an involution $*: H_*(M_g;\mathbb{Q}) \to H_*(M_g;\mathbb{Q})$ by the formula

$$a_i^* = b_i, \quad b_j^* = a_j, \quad \mathbb{I}^* = \mathbb{T}, \quad \mathbb{T}^* = \mathbb{I}, \tag{11}$$

i.e., the involution $*$ is up to sign, the Poincaré duality map. For a given word $w = w_1 w_2 \cdots w_{2n}$ let $w^* = w_1^* w_2^* \cdots w_{2n}^*$. Note that the first part of the following proposition is just a reformulation of Lemma 1.11, while the second part gives a precise formula for the sign function $\epsilon_{w,w'}$.

Proposition 1.12:

$$B(w, w') \neq 0 \Longrightarrow w' = w^*,$$
$$B(w, w^*) = (-1)^{\binom{\alpha(w)}{2}} \langle w_1, w_1^* \rangle \cdots \langle w_{2n}, w_{2n}^* \rangle = (-1)^{\binom{\alpha(w)}{2} + \beta(w)},$$

where $\beta(w)$ is the number of occurrences of letters $b_1, \ldots, b_g$ in w, while $\alpha(w)$ is the number of occurrences of both a_i and b_j in w. If $\mathbb{I}$ and $\mathbb{T}$ do not appear in w whatsoever, then $B(w, w^*) = +1$.

Let $\pi \in S_{2n}$ be a permutation and let $\alpha(\pi) = 1^{\alpha_1} 2^{\alpha_2} \cdots (2n)^{\alpha_{2n}}$ be the associated partition of $[2n] = \{1, 2, \ldots, 2n\}$. In light of the well known equality $\mathrm{Sign}(g \times h, M \times N) = \mathrm{Sign}(g, M)\,\mathrm{Sign}(h, N)$, [4], [9],

$$\mathrm{Sign}(\pi, (M_g)^{2n}) = \prod_{k=1}^{2n} \{\mathrm{Sign}(C_k, (M_g)^k)\}^{\alpha_k} \tag{12}$$

where $C_k: (M_g)^k \to (M_g)^k$ is a cyclic permutation of coordinates,

$$C_k(x_1, x_2, \ldots, x_k) = (x_2, \ldots, x_k, x_1).$$

Proposition 1.13:

$$\mathrm{Sign}(C_k, (M_g)^k) = \begin{cases} 2 - 2g, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Proof: Let $V = H_k(M_g) \otimes \mathbb{C}$ and let $B: V \times V \to \mathbb{C}$ be the hermitian form obtained by formula (8) from the intersection form on $(M_g)^k$. Let $\omega: V \to V$ be the map induced by C_k. As before, $\omega: V \to V$ is a B-preserving endomorphism. If V admits a B-orthogonal decomposition of the form $V = V_1 \oplus V_2$ where V_1 and V_2 are ω-invariant subspaces, then

$$\mathrm{Sign}(\omega, V) = \mathrm{Sign}(\omega_1, V_1) + \mathrm{Sign}(\omega_2, V_2) \tag{13}$$

where $\omega_i := \omega|V_i$ is the restriction of ω on V_i. Given a word $w = w_1 \cdots w_k$, let $V_w = \mathrm{span}\{w, \omega(w), \ldots, \omega^{k-1}(w)\}$ be the minimal ω-invariant subspace of V which contains vector w. Let $W_w := V_w + V_{w^*}$. Then by Proposition 1.12, for any two words w and w', the associated spaces W_w and $W_{w'}$ are either identical or mutually orthogonal. By formula (13), it suffices to find $\mathrm{Sign}(\omega, W_w)$. If $V_w \neq V_{w^*}$ then $\mathrm{Sign}(\omega, W_w) = 0$, so we assume that $W_w = V_w$. Note that $\dim(V_w) = p$, where $p = p(w) := \min\{s \geq 1 \mid \omega^s(w) = \pm w\}$. If k is odd, then p, being a divisor of k, must be also an odd number. This implies that $w^* \neq \pm\omega^j(w)$ for all j, which means that the form $B|V_w$ is zero and $\mathrm{Sign}(\omega, V_w) = 0$. It immediately follows that $\mathrm{Sign}(C_k, (M_g)^k) = 0$ if k is an odd number. If $k = 2m$ is even, then a nonzero contribution from V_w can be expected only if p is an even number. In this case we can apply Proposition 1.10 and deduce that $p = 2$. The list of all basic words u having the property $p(u) = 2$ is

$$w = \mathbb{I}\mathbb{T} \cdots \mathbb{I}\mathbb{T}, \quad w_i := a_i b_i \cdots a_i b_i, \quad i = 1, \ldots, g.$$

Since $\mathbb{I}$ and $\mathbb{T}$ are classes of even degree, $\omega(w) = w^*$ and $B(w, \omega(w)) = +1$. Since a_i, b_i are classes of degree 1, $B(w_i, \omega(w_i)) = (-1)^{2m-1} B(w_i, w_i^*)$, and knowing that by Proposition 1.12, $B(w_i, w_i^*) = (-1)^{\beta(w_i) + m(2m-1)} = +1$, we have $B(w_i, \omega(w_i)) = -1$. Finally, by Proposition 1.10,

$$\mathrm{Sign}(C_{2m}, (M_g)^{2m}) = \mathrm{Sign}(V_w) + \sum_{i=1}^{g} \mathrm{Sign}(V_{w_i}) = 2 - 2g. \qquad \blacksquare$$

PROPOSITION 1.14: *If $s \geq 1$ then*

$$\mathrm{Sign}(C_k, (M_{g,s})^k) = \begin{cases} -2g, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Proof: As in the proof of Proposition 1.13, we need information about the intersection pairing

$$B\colon H_{2n}(M_{g,k}^{2n}; \mathbb{Q}) \times H_{2n}(M_{g,k}^{2n}; \mathbb{Q}) \to \mathbb{Q}.$$

The homology group $H_1(M_{g,s}, \mathbb{Z}) \cong \mathbb{Z}^{2g+s-1}$ has a basis $e_1, e_2, \ldots, e_{2g+s-1}$ where $e_i = a_i, e_{g+j} = b_j$ for $i = 1, \ldots, g$ and e_j for $j \geq 2g+1$ correspond to the holes $\alpha_1, \ldots, \alpha_{s-1}$ in M_g. The group $H_{2n}(M_{g,s}^{2n}; \mathbb{Q})$ is generated by the classes (words) of the form $w = w_1 \times \cdots \times w_{2n} = w_1 \cdots w_{2n}$ where $w_i \in \{e_1, \ldots, e_{2g+s-1}\}$. Lemma 1.11 is still true with the simplification that both $\mathbb{I}, \mathbb{T}$ and the classes $\{e_j\}_{j=2g+1}^{2g+s-1}$, associated to the holes in $M_{g,s}$, are excluded. The rest of the proof follows the argument of the proof of Proposition 1.13. $\blacksquare$

Proof of Theorem 1.8: As already observed, Theorem 1.8 is trivially correct if m is an odd number since in that case both sides of equation (7) are zero. Let us assume that m is an even number, $m = 2n$. By equation (12) and Proposition 1.13,

$$
\begin{aligned}
\mathrm{Sign}(SP_G^{2n}(M_g)) &= \frac{1}{(2n)!}\sum_{\pi\in G}\mathrm{Sign}(\pi,(M_g)^{2n}) \\
&= \frac{1}{(2n)!}\sum_{\pi\in G}\prod_{k=1}^{2n}\{\mathrm{Sign}(C_k,(M_g)^k)\}^{\alpha_k(\pi)} \qquad (14)\\
&= Z(G;0,2-2g,\ldots,0,2-2g)
\end{aligned}
$$

which establishes part (a) of Theorem 1.8. Part (b) follows by the same argument from Proposition 1.14. ∎

1.2 Corollaries of Theorem 1.2. We start with the following elementary lemma.

Lemma 1.15:

$$Z(S_n;\alpha,\beta,\alpha,\beta,\ldots) = [t^n](1-t)^{-\frac{1}{2}(\alpha+\beta)}(1+t)^{\frac{1}{2}(\alpha-\beta)}. \tag{15}$$

Proof: The result is easily deduced from the well known fact [1], [11], [3], [14]

$$Z(S_n;x_1,\ldots,x_n) = [t^n]\exp\Big(x_1t+\frac{x_2t^2}{2}+\frac{x_3t^3}{3}+\cdots\Big). \tag{16}$$

In particular, for $\alpha=\beta$ one has

$$Z(S_n;\alpha,\alpha,\ldots) = [t^n](1-t)^{-\alpha} = (-1)^n\binom{-\alpha}{n} \tag{17}$$

while for $\alpha = 0$ formula (15) reduces to

$$Z(S_n;0,\beta,0,\beta,\ldots) = [t^n](1-t^2)^{-\beta/2}. \tag{18}$$

Theorem 1.16 (G. Pólya, [1]): *Let $G = S_k \wr H$ be the wreath product of S_k by a subgroup $H \subset S_m$ of the symmetric group S_m. Then the cycle index $Z(G;\vec{x})$ of G can be computed from the cycle indices of S_k and H by the formula*

$$Z(G;\vec{x}) = Z(S_k;Z(H;x_1,x_2,\ldots),Z(H;x_2,x_4,\ldots),\ldots,Z(H;x_k,x_{2k},\ldots)). \tag{19}$$

Proof of Proposition 1.4: By Theorems 1.2 and 1.16,

$$\begin{aligned}\mathrm{Sign}(SP_{S_p\wr S_m}(M_{g,k})) &= Z(S_p\wr S_m;0,-2g,0,-2g,\dots)\\ &= Z(S_p;Z(S_m;0,-2g,0,-2g,\dots),Z(S_m;-2g,-2g,\dots),\dots)\\ &= Z(S_p;0,(-1)^m\binom{2g}{m},0,(-1)^m\binom{2g}{m},\dots) = (-1)^{p/2}\binom{\frac{1}{2}\binom{2g}{m}}{\frac{1}{2}p}.\end{aligned}$$ ∎

1.3 Non-homeomorphic symmetric products.

Proof of Theorem 1.1: Since $M_{g,k}$ is, up to homotopy, a wedge of $2g+k-1$ circles, the condition $2g+k=2g'+k'$ implies $M_{g,k}\simeq M_{g',k'}$ and as a consequence $SP^m(M_{g,k})\simeq SP^m(M_{g',k'})$. Suppose that m is an even integer, $m=2n$. The open manifold $SP^{2n}(M_{g,k})$, according to Corollary 1.3, has signature $(-1)^n\binom{g}{n}$. The condition $\max\{g,g'\}\geq n$ guarantees that either $\binom{g}{n}$ or $\binom{g'}{n}$ is nonzero. The sequence $\binom{g}{n}$, as a function of g, is strictly monotone for $g\geq n$. Together with the condition $g\neq g'$ this implies

$$\mathrm{Sign}(SP^{2n}(M_{g,k}))\neq \mathrm{Sign}(SP^{2n}(M_{g',k'})),$$

hence $SP^{2n}(M_{g,k})$ and $SP^{2n}(M_{g',k'})$ are not homeomorphic. The case of an odd integer m is treated similarly. If, contrary to the claim, $SP^m(M_{g,k})$ and $SP^m(M_{g',k'})$ are homeomorphic, then by Corollary 1.5, $\binom{2g}{m}=\binom{2g'}{m}$. This again would contradict the conditions $\max\{g,g'\}\geq m/2$ and $g\neq g'$. ∎

References

[1] M. Aigner, *Combinatorial Theory*, Springer-Verlag, Berlin, 1979.

[2] A. Björner, *Topological methods*, in *Handbook of Combinatorics* (R. Graham, M. Grötschel and L. Lovász, eds.), North-Holland, Amsterdam, 1995.

[3] I. M. Gessel and R. P. Stanley, *Algebraic enumeration*, in *Handbook of Combinatorics II* (R. L. Graham, M. Grötschel and L. Lovász, eds.), North-Holland, Amsterdam, 1995.

[4] C. M. Gordon, *The G-signature theorem in dimension* 4, in *A la Recherche de la Topologie Perdue* (L. Guillou and A. Marin, eds.), Progress in Mathematics 62, Birkhäuser, Boston, 1986.

[5] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.

[6] A. Grothendieck, *Sur quelques points d'algebre homologique*, Tôhoku Mathematical Journal (2) **9** (1957), 119–221.

[7] F. Hirzebruch, *The signature theorem: reminiscences and recreation*, in *Prospects in Mathematics*, Annals of Mathematics Studies 70, Princeton University Press, Princeton, 1971, pp. 3–31.

[8] F. Hirzebruch, *Lectures on the Atiyah–Singer Theorem and its Applications*, Unpublished lecture notes, Berkeley, 1968.

[9] F. Hirzebruch and D. Zagier, *The Atiyah–Singer Theorem and Elementary Number Theory*, Publish or Perish, Inc., Boston, 1974.

[10] S. Kallel, *Divisor spaces on punctured Riemann surfaces*, Transaction of the American Mathematical Society **350** (1998), 135–164.

[11] V. Krishnamurthy, *Combinatorics, Theory and Applications*, East–West Press, New Delhi, 1985.

[12] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proceedings of the Cambridge Philosophical Society **58** (1962), 563–568.

[13] I. G. Macdonald, *Symmetric product of an algebraic curve*, Topology **1** (1962), 319–343.

[14] R. Stanley, *Enumerative Combinatorics*, Vols. I and II, Studies in Advanced Mathematics No. 49, Cambridge University Press, Cambridge, 2001.

[15] K. Trenčevski and D. Dimovski, *Complex Commutative Vector Valued Groups*, Macedonian Academy of Science and Arts, Skopje, 1992.

[16] K. Trenčevski and D. Dimovski, *On the affine and projective commutative $(m+k,m)$-groups*, Journal of Algebra **240** (2001), 338–365.

[17] D. B. Zagier, *Equivariant Pontrjagin Classes and Applications to Orbit Spaces*, Lecture Notes in Mathematics **290**, Springer-Verlag, Berlin, 1972.